Dynamical relaxation of dark energy: A solution to early inflation, late-time acceleration and the cosmological constant problem

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In recent years different explanations are provided for both an inflation and a recent acceleration in the expansion of the universe. In this Letter we show that a model of physical interest is the modification of general relativity with a Gauss-Bonnet term coupled to a dynamical scalar-field as predicted by certain versions of string theory. This construction provides a model of evolving dark energy that naturally explains a dynamical relaxation of the vacuum energy (gravitationally repulsive pressure) to a small value (exponentially close to zero) after a sufficient number of e-folds. The model also leads to a small deviation from the $w = -1$ prediction of non-evolving dark energy.

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Inflation, or a period of accelerated expansion in the early universe, and cosmic acceleration at late times as pure gravitational dynamics are unusual within the context of general relativity (GR). Einstein back in 1917 amended his General Theory of Relativity with a cosmological constant Λ to achieve a stationary universe, but it was later realized that a positive Λ, in a cosmological context, gives rise to an accelerating expansion of the universe. In modern parlance, a positive Λ, more generally, a vacuum energy, is called "dark energy", which makes up about 70% of the matter–energy content of the present universe. The right explanation of dark energy is the greatest challenge faced by the current generation of physicists and cosmologists.

A pure cosmological constant, $\Lambda \sim (10^{-3}eV)^4$, as a source of dark energy does not seem much likely, as there is no conceivable mechanism to explain why it is so tiny and also why should it be comparable to the present matter density [1] (see the reviews [2]). It is very problematic for particle physics if Λ is to be interpreted as the vacuum energy. If dark energy is something fundamental, then it is natural to attribute it to one or more scalar fields [3]. It is not only because dark energy associated with the dynamics of scalar fields which are uniform in space provides a mechanism for generating the observed density perturbations [4] and a negative pressure sufficient to drive the accelerating expansion [5], but there is another reason well motivated by fundamental physics.

Gravity is attractive and thus curves spacetime whose dynamics is set entirely by the spacetime curvature: the Riemann curvature tensor. However, inflation (or cosmic acceleration) is repulsive: this is caused not by gravity altering its sense, attraction to repulsion, but due to an extra source, or vacuum energy, which covers every point in space and exerts gravitationally smooth and negative pressure. The fundamental scalar fields abundant in higher dimensional theories of gravity and fields, such as string/M theory, are such examples. Most of the scalars in string theory are the structure moduli associated with the internal geometry of space, which do not directly couple with the curvature tensor. But a scalar field associated with the overall size and the shape of the internal compactification manifold generically couples with Riemann curvature tensor, even in four dimensions [6].

The question that naturally arises is: what is the most plausible form of a four-dimensional gravitational action that offers a resolution to the dilemma posed by the current cosmic acceleration [7], or the dark energy problem, within a natural theoretical framework? In recent years, this problem has been addressed in hundreds of papers proposing various kinds of modification of the energy-momentum tensor in the "vacuum" (i.e., the source of a gravitationally repulsive pressure), e.g., phantom field [8], $f(R)$ gravity that adds terms proportional to inverse powers of R [9] or the R^2 terms, or both these effects at once [10], braneworld modifications of Einstein's GR [11], etc.. Other examples of recent interest are cosmological compactifications of string/M theory on a hyperbolic manifold [12] and on twisted spaces [13].

A very desirable feature of a theory of scalar-tensor gravity is quasi-linearity: the property that the highest derivatives of the metric appear in the field equations only linearly, so as to make the theory ghost free. Interestingly, differential geometry offers a particular combination of the curvature squared terms with such behav-

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ior, known as the Gauss-Bonnet (GB) integrand:

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}.$$

All versions of string theory in 10 dimensions (except type II) include this term as the leading order α' correction [14]. A unique property of the string effective action is that the couplings are field dependent, and thus in principle space-time dependent. The effective action for the system may be taken to be

$$S = \int d^dx\sqrt{-g}\left[\frac{R}{2\kappa^2} - \frac{\gamma}{2}(\nabla\sigma)^2 - V(\sigma) + f(\sigma)\mathcal{G}\right], \quad (1)$$

where κ is the inverse Planck mass $M_P^{-1} = (8\pi G_N)^{1/2}$, γ is a coupling constant and $f(\sigma) = \lambda - \hat{\delta}\xi(\sigma)$: the coupling λ may be related to string coupling g_s via $\lambda \sim 1/g_s^2$. The numerical coefficient $\hat{\delta}$ typically depends on the massless spectrum of every particular model [15]. $V(\sigma)$ is phenomenologically-motivated field potential. Our approach is a new take on a familiar system: the Gauss-Bonnet combination is multiplied by a function of the scalar field. In four dimensions ($d = 4$) the GB term makes no contribution if $f(\sigma) = \text{const}$. It is natural to consider $f(\sigma)$ as a dynamical variable. This follows, for example, from the one-loop corrected string effective action [6] when going from the string frame to the Einstein frame to appropriately describe the universe we observe today, where the equivalence principle is well preserved and the Newton's constant G_N is (almost) time-independent. A discussion on cosmological advantages of modified Gauss-Bonnet theory may be found in [18]. A study has been recently made in [19] by introducing higher order (quartic) curvature corrections, but in a background of fixed modulus field σ.

The most pertinent question that we would like to ask is: what new features would a dynamical GB coupling introduce, and how can they influence a cosmological solution? In this Letter we present an exact cosmological solution that explains a dynamical relaxation of vacuum energy to a small value (exponentially close to zero) after a sufficient number of e-folds, leading to a small deviation from the $w = -1$ prediction of non-evolving dark energy.

Let us consider the four-dimensional spacetime metric in standard Friedmann-Robertson-Walker (FRW) form:

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}(dx^i)^2, \quad (2)$$

where $a(t)$ is the scale factor of the universe. In terms of the following dimensionless variables

$$\begin{aligned} &x = (\gamma\kappa^2/2)(\dot{\sigma}/H)^2, \qquad && y = \kappa^2(V(\sigma)/H^2), \\ &u = 8\kappa^2 f(\sigma)H^2, && h = \dot{H}/H^2, \end{aligned} \quad (3)$$

the Einstein field equations obtained by varying the metric $g_{\mu\nu}$ are given by

$$\begin{aligned} 0 &= -3 + x + y - 3(u' - 2hu), &(4)\\ 0 &= u'' + (2-h)u' - 2h'u - 2(2+h)uh \\ &\quad + 2h + 3 + x - y, &(5) \end{aligned}$$

where $X' = \frac{\mathrm{d}X}{\mathrm{d}\eta} = \frac{1}{H}\frac{\mathrm{d}X}{\mathrm{d}t} = a\frac{\mathrm{d}X}{\mathrm{d}a}$, $\eta = \int H\,\mathrm{d}t = \ln(a/a_0)$ is the number of e-folds and $H = \frac{\dot{a}}{a}$ is the Hubble expansion parameter. The time evolution of σ is given by

$$\frac{d}{dt}\left(\gamma\,\dot{\sigma}^2 + 2\Lambda(\sigma)\right) = -6H\left(\gamma\dot{\sigma}^2\right) - 2\delta, \quad (6)$$

where $\Lambda(\sigma) \equiv V(\sigma) - f(\sigma)\mathcal{G}$ and $\delta \equiv f(\sigma)\frac{d\mathcal{G}}{dt}$, with $\mathcal{G} = 24H^2(\dot{H} + H^2)$. We will call $\Lambda(\sigma)$ an effective potential. Eq. (6) may be written as $x' + 2(h+3)x + y' + 2hy - 3(h+1)(u' - 2hu) = 0$.

When $f(\sigma) = 0$ (or $u = 0$), the equations of motion satisfy simple relationships: $y = 3+h$ and $x = -h$. From this we see that it is enough to introduce only one new parameter (e.g. the potential $V(\sigma)$) to explain a particular value of slow-roll type variable h ($\equiv a\ddot{a}/\dot{a}^2 - 1$) or the acceleration parameter $\ddot{a}$; if more than one parameter is introduced in the starting action, as such the case with $f(\sigma) \neq 0$, then one complicates the problem. This remark though valid to some extend may have less significance for at least two reasons. Firstly, a non-trivial coupling between the field σ and the curvature squared terms naturally exists in one-loop corrected string effective action [15], as well as in the best motivated scalar-tensor theories which respect most of GR's symmetries, see for example [16, 17]. Secondly, the knowledge about the modulus-dependent GB coupling may help one to construct a cosmological model where h (or the dark energy equation of state parameter, w) is dynamical.

While there can be a myriad of scalar field models with different forms of $V(\sigma)$ and $f(\sigma)$, inspired by particle physics beyond the standard model or string theory, not all scalar potentials may be used to describe the universe that we observe today. In order for the model to work a scalar field should relax its potential energy after inflation down to a sufficiently low value, presumably very close to the observed value of the dark energy in order to solve the cosmological constant problem. In this Letter, rather than picking up particular functional forms for $V(\sigma)$ and $f(\sigma)$, for example, as in Ref. [20], we would like to obtain an exact cosmological solution, which respects the symmetry of the field equations.

By solving the field equations (4)-(6), we can write x and y (and hence $\Lambda(\sigma)$) as a second order differential equation in u. Notably, $\kappa^2\Lambda(\sigma) = \frac{H^2}{2}\left[u'' + (5-h)\,u' - 2(8h + h^2 + h' + 3)u\right] + H^2(3+h)$, which generally has a solution composed of *homogeneous*

and *nonhomogeneous* parts. We refer to the solution of $u''+(5-h)\,u'-2(8h+h^2+h'+3)u=0$, or equivalently,

$$\kappa^2\Lambda(\sigma)=H^2(3+h)=3H^2+HH', \qquad (7)$$

a *homogeneous* solution, which is trivially satisfied for $f(\sigma)=0$. Effectively, for $f(\sigma)\neq 0$, we will pick $V(\sigma)=f(\sigma)\mathcal{G}+H^2(3+h)/\kappa^2$, leaving us with only one arbitrary function, out of $x(\eta)$, $h(\eta)$ or $u(\eta)$.

What is the advantage of adding a new term, $f(\sigma)\mathcal{G}$, if one could make an ansatz like (7) with $f(\sigma)=0$? An interesting feature of our construction is that while the contributions coming from both the field potential $V(\sigma)$ and the coupled GB term, $f(\sigma)G$, can be large separately, the effective potential, $\Lambda(\sigma)$, can be exponentially close to zero at late times, as it relaxes to a small value after a sufficiently large number of e-folds of expansion. Moreover, even if we do not impose *a priori* the constraint $V(\sigma)-f(\sigma)\mathcal{G}=H^2(3+h)/\kappa^2$ we arrive at a similar expression for various other solutions; below we will present one such example.

Given a perfect fluid form for the energy momentum tensor of the field σ: $T_{00}=\rho_\sigma$, $T_{ii}=p_\sigma a(t)^2$, we define the equation of state (EOS) parameter

$$w\equiv\frac{p_\sigma}{\rho_\sigma}=-\frac{2h+3}{3}=\frac{2q-1}{3}, \qquad (8)$$

where $q\equiv -a\ddot{a}/\dot{a}^2$ is the deceleration parameter. We will assume first that we are dealing with a canonical-scalar ($\gamma>0$), so that $1+w\geq 0$ (or $h\leq 0$). When $h\simeq 0$, and hence $H\simeq \text{const}=H_{\text{inf}}$, $\Lambda(\sigma)$ acts as a cosmological constant term, i.e. $\Lambda(\sigma)\simeq 3M_p^2H_{\text{inf}}^2\equiv\Lambda_{\text{inf}}$. A major difference from the case of a cosmological constant Λ_0, for which $p_\Lambda=-\rho_\Lambda$ and thus $w_\Lambda=-1$, is that the field is evolving towards an analytic minimum at $\Lambda(\sigma)\simeq 0$.

The Gauss-Bonnet term, $\mathcal{G}=24\frac{\dot{a}^2\ddot{a}}{a^3}$, changes sign between accelerating ($\ddot{a}>0$) and decelerating ($\ddot{a}<0$) solutions. In our model, the effective potential $\Lambda(\sigma)$ ($\equiv V(\sigma)-f(\sigma)\mathcal{G}$) has to be non-negative in order that the vacuum becomes a possible ground state. Under the condition (7), it is required that $h\geq -3$ and consequently the EOS parameter $w\leq 1$. This last inequality may be saturated only by stiff matter for which $w=1$ (and the velocity of sound approaches the velocity of light). In general, however, $w<1$ holds and hence $\Lambda(\sigma)>0$.

Under the condition (7) alone, the field potential $V(\sigma)$ may not always be positive. Then one may ask: Is there any condition for the (semi-)positivity of the potential? Below we will give the condition for $V(\sigma)\geq 0$, though it may not be required for obtaining inflationary type solutions in the present context. Furthermore the condition (7) fixes only one out of two arbitrary parameters in the model. The last arbitrary parameter may be fixed either by allowing one of the field variables to take a fixed (but arbitrary) value or by making an appropriate ansatz for one of the field variables. Here we consider two well motivated examples.

If we use the approximation $h\equiv\dot{H}/H^2=H'/H\simeq \text{const}=h_0$, then, by solving (7), we find

$$H=\mathrm{e}^{\int h\mathrm{d}\eta}=H_0\mathrm{e}^{h_0\eta},\quad u=u_1\,\mathrm{e}^{\alpha-\eta}+u_2\,\mathrm{e}^{\alpha+\eta}, \qquad (9)$$

where H_0 and u_i are integration constants, and

$$\alpha_\pm=\frac{1}{2}\left(h_0-5\pm\sqrt{9h_0^2+54h_0+49}\right). \qquad (10)$$

It is the coupling α_+ (α_-) which is of greater interest for $\eta\gtrsim 0$ ($\eta\lesssim 0$). The scale factor is given by $a(t)=a_0(1-c_0h_0t)^{-1/h_0}$, where $c_0h_0<0$, implying that the universe accelerates in the range

$$-1<h_0<0\;\Rightarrow\;0>q>-1. \qquad (11)$$

We then wish to consider the case where h is dynamical. To this aim, we make no assumptions about the form of h, and instead consider the ansatz that during a given epoch we may make the approximation

$$u\equiv 8\kappa^2f(\sigma)H^2\approx u_{time}e^{\alpha_{time}\eta} \qquad (12)$$

where the *time* can be *early* or *late*, where $|u_{early}|>|u_{late}|$, and $\alpha_{early}<\alpha_{late}$. This *ansatz* is well motivated, since in the context of string theory the coupling $f(\sigma)$ is generally a sum of exponential terms (as functions of σ or the number of e-folds) and so is the Hubble parameter. Interestingly, the ansatz (12) for the form of u will allow us to write h in a closed form:

$$H\;=\;H_0\,\cosh\beta(\eta-\eta_1)\,\mathrm{e}^{-\hat{\beta}\eta}, \qquad (13)$$

$$h\;=\;-\hat{\beta}+\beta\tanh\beta(\eta-\eta_1), \qquad (14)$$

where η_1 is a free parameter,

$$\hat{\beta}\equiv 4+\frac{\alpha}{4},\quad \beta\equiv\frac{1}{4}\sqrt{9\alpha^2+72\alpha+208}, \qquad (15)$$

and $\alpha=\alpha_{\text{early}}$ or α_{late}. Writing the expressions for $x(\eta)$ and $y(\eta)$ is straightforward. Analogous to an inflationary type solution induced by a conformal-anomaly [21], the solutions given above are singularity-free. Note, for $\Delta\eta\equiv\eta-\eta_1\gtrsim 0$, we always have $\Lambda(\sigma)>0$ for $\alpha\leq 1$ (cf Fig. 1).

Next, consider that $u(\sigma)=8\kappa^2f(\sigma)H^2\simeq\text{const}\equiv 1/\hat{\alpha}$ and $x\equiv\frac{(\gamma/2)\dot{\sigma}^2}{H^2}\kappa^2\equiv x_0$, where $|\hat{\alpha}|>1$ and $0<x_0<1$, which are reasonably good approximations at low energy. In this case, $\eta=\pm\sqrt{\gamma/2x_0}\,(\sigma/M_{Pl})$+const. The solution is again given by (13)-(14) but now

$$\hat{\beta}=-\frac{1+\hat{\alpha}}{2},\quad \beta=\frac{1}{2}\sqrt{(1+\hat{\alpha})^2+4x_0\hat{\alpha}}. \qquad (16)$$

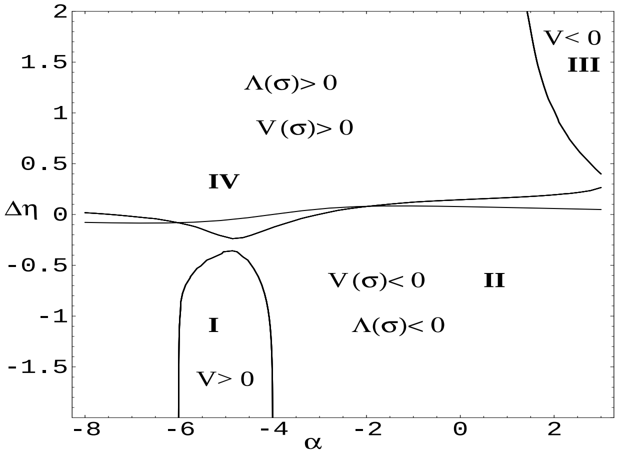

FIG. 1: The $\Delta\eta$ - α phase space, showing regions with $V(\sigma) > 0$ and $V(\sigma) < 0$. The almost horizontal line with $h = -3$ separates the regions between $\Lambda(\sigma) < 0$ and $\Lambda(\sigma) > 0$. Here we have set $u_{time} = 0.2$. For a large value of u_{time} (> 0.2), the regions I and IV, both with $V(\sigma) > 0$, merge to a single region, while for a small value of u_{time} (< 0.2), the region I moves to a more negative value of $\Delta\eta$, while the region III moves to a more positive value of $\Delta\eta$ (and α). In any case, the region with $\Lambda(\sigma) < 0$ (or $h+3 < 0$) is physically less relevant as it corresponds to the case where the EOS parameter $w > 1$.

In particular, for $x_0 \ll 1/(4\hat{\alpha})$, or $\beta \simeq -\hat{\beta}$, one has $H \propto H_0[1 + e^{2\beta(\eta-\eta_1)}]$; the Hubble expansion rate decreases with η when $\beta(\eta - \eta_1) < 0$. The scalar potential and the coupled GB term may be given by

$$V(\sigma) = M_P^2 H^2 \left(-\frac{3}{\hat{\alpha}} - \frac{6\beta}{\hat{\alpha}} \tanh \beta\tilde{\sigma} - x_0\right), \quad (17)$$

$$f(\sigma)\mathcal{G} = M_P^2 H^2 \left(\frac{3+\hat{\alpha}}{16\hat{\alpha}} + \frac{\beta}{8\hat{\alpha}} \tanh \beta\tilde{\sigma}\right), \quad (18)$$

where $\tilde{\sigma} = \sqrt{\gamma/2x_0}\,\kappa(\sigma - \sigma_1)$. Note that here σ (as well as $\tilde{\sigma}$) can take a negative value, so that $V(\sigma) > 0$ for $3 + x_0\hat{\alpha} < 6\beta$, or, more precisely, for $h < (3 - x_0)\hat{\alpha}/6$. Initially, $V(\sigma) - f(\sigma)\mathcal{G} \neq M_P^2 H^2(3 + h)$, but after a certain number of e-folds, namely $|\beta(\eta - \eta_1)| > 2$, so that $|\tanh \beta\tilde{\sigma}| \to 1$, the relation as above (or Eq. (7)) may be attained by satisfying $(1 + 8x_0\hat{\alpha}) = (49 + 8\hat{\alpha})(\beta - \hat{\beta})$. That is, the condition like (7) puts an additional constraint on the model parameters but it may not exhaust the basic characteristics of the model.

A question may be raised as to whether inflation is due to the new term in the action, $f(\sigma)\mathcal{G}$, or simply due to the potential $V(\sigma)$. In string theory context, the coupling $f(\sigma)$ may be expanded as $f(\sigma) \propto \left[\frac{2\pi}{3}\cosh(\sigma) - \ln(2)\right] + \text{const.}$ At early times, $\sigma < 0$, the potential $V(\sigma)$ is expected to dominate the Gauss-Bonnet contribution, $f(\sigma)\mathcal{G}$. At the start of inflation, $\mathcal{G} = 24\left(\frac{\dot{a}}{a}\right)^2 \frac{\ddot{a}}{a} \to 0$. Inflation is mainly due to the potential which is not essentially flat; in such a case the Hubble expansion rate could naturally drop to a significantly low value after a sufficient number of e-folds. This can be understood also from some earlier studies on the subject [15, 23]. With $V(\sigma) = 0$, the period of acceleration is short and the number of e-folds is order of unity; a model with vanishing potential may not be suitable for early inflation. At late times, the potential drops relatively faster as compared to the coupled Gauss-Bonnet term, so the contribution of the potential nearly equates or slightly exceeds the contribution of the GB term.

Let us discuss the result (13)-(14) in some details. The universe starts to expand as η increases from η_i, but it accelerates only when $\eta \gtrsim \eta_1$. Eventually when $\eta \gtrsim \eta_1 + 2.5/\beta$, the scalar field begins to freeze in, so that $w \leq -1/3$; the actual value of w depends on the value of α ($= \alpha_{\rm early}$), see Figs. 2 and 3. After a certain number of e-folds, say N, our approximation, (12), with the *time* being *early* breaks down. Sometime later, subsequent evolution will be controlled by (12) with the *time* being *late*. For $\eta \lesssim \eta_{late}$ the universe is in a deceleration phase which implies that inflation must have stopped during the intermediate epoch. As η crosses η_{late}, the universe begins to accelerate for the second time.

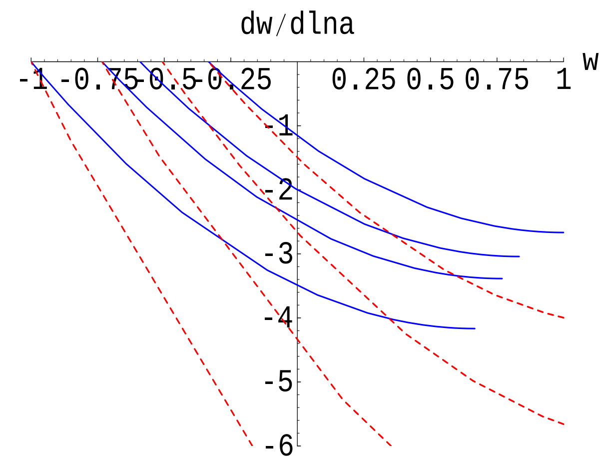

FIG. 2: The $w - w'$ phase space occupied by the scalar field, in the range $0 \geq \eta - \eta_1 \geq 5$. From left to right $\alpha = -6, -5.38, -5, -4$ (solid lines) and $\alpha = +1, -0.0148, -1, -2$ (dashed lines). The universe is not accelerating for $-4 < \alpha < -2$. The variation of w' w.r.t. w is significant in the range $|\eta - \eta_1| \lesssim 2.5/\beta$. This happens when the red-shift factor $z < 0.8$ (2.49), since $\beta < 4.25$ (> 2), if $|\eta - \eta_1|$ is to be related to z via $1 + z = e^{\eta - \eta_1}$. For $|\eta - \eta_1| \gtrsim 2.5/\beta$, the field is almost frozen, $w \simeq$ const; $w \leq -1/3$ if $\eta > \eta_1$, while $w > 0$ if $\eta < \eta_1$.

A large shift in Hubble rate may be required to explain relaxation of vacuum energy from a natural value $\Lambda(\sigma) \sim M_P^2 H_b^2 \sim 10^{-8} M_P^4$ (*before* inflation) to a sufficiently low value $\Lambda(\sigma) \sim M_P^2 H_a^2 \sim 10^{-120} M_P^4$ (*after* inflation), viz $H_b \sim 10^{23}\,$eV and $H_a \sim 10^{-33}\,$eV. One thus calculates the change in H during an inflationary epoch of N e-

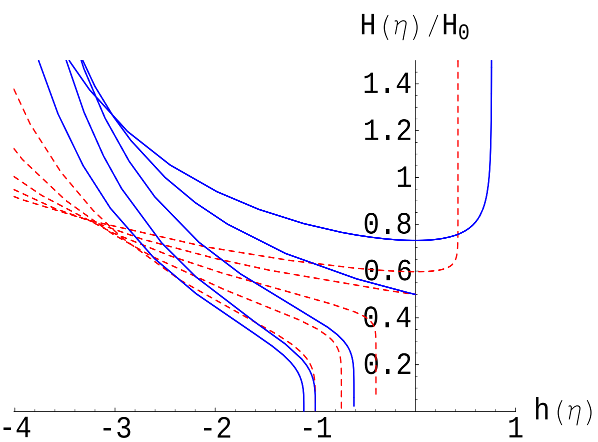

FIG. 3: The $h-H$ phase space occupied by the scalar field, in the range $-5 \le \eta - \eta_1 \le 5$. From top to bottom $\alpha = -7, -6, -5, -4, -3$ (solid lines) and $\alpha = 2, 1, 0, -1, -2$ (dashed lines). For a canonical scalar (i.e. $\gamma > 0$), α is only well defined for $-6 < \alpha < 1$. The Hubble parameter decreases with proper time for the coupling $-6 < \alpha < 1$, while it increases with proper time for the coupling $\alpha < -6$ or $\alpha > 1$, heading to a value $w < -1$, since $h > 0$. In this last case, we get a negative kinetic energy for the field σ, in some regions of field space, which then requires $\gamma < 0$ (phantom field).

folds, by considering the ratio

$$\varepsilon \equiv \frac{H(\eta_i + N)}{H(\eta_i)} = \frac{\cosh\beta(\Delta\eta + N)\,\mathrm{e}^{-\hat{\beta}N}}{\cosh\beta(\Delta\eta)}, \tag{19}$$

where $\Delta\eta \equiv \eta_i - \eta_1$, with η_i being the time at on-set of inflation. In our model, the total number of e-folds N required to get a small ratio, like $\varepsilon \sim 10^{-56}$, depends on the value of α, which is related to the slope of the potential coming purely from the Gauss-Bonnet coupling, $f(\sigma)\,\mathcal{G}$. The value of N would be small, $N \sim 129$, for $\alpha \gtrsim -2$ (or $\alpha \lesssim -4$), while it would be large for $\alpha \lesssim 1$ (or $\alpha \gtrsim -6$), $N \sim \mathcal{O}(300)$.

The solution (9)-(14) may also be used to explain a cosmic acceleration of the universe at late times, i.e., at an energy scale many orders of magnitude lower from inflation. The present value of the parameter h must be determined by observation. For $|\eta - \eta_1| \gtrsim 2.5/\beta$, $h(\eta)$ varies with the logarithmic time, η, almost negligibly. The equation of state parameter for the dark energy is $w = -(2h+3)/3$. From this, and using (14), we get $w \ge -1$ and $\hat{\beta} \ge \beta$ for $-6 \le \alpha \le 1$, so that H decreases with proper time (Fig. 3). For instance, if the observed value of the deceleration parameter corresponds to

$$q_{\rm obs} = -1 - h_{\rm obs} \simeq -\hat{\beta} - \beta \simeq -0.6, \tag{20}$$

then $\alpha = -5.3851$ or -0.0148 and hence $w \simeq -0.73$. One may achieve $w < -1$, for $\eta \gg \eta_1 + 2.5/\beta$, by allowing $\alpha < -6$ or $\alpha > 1$. In this case, however, since $\beta > \hat{\beta}$, H increases with proper time. There are arguments that the present cosmological data may even favor a value $w < -1$[22]; if this is the case, in our model, one has to allow $\alpha > 1$ or $\alpha < -6$.

The equation of state parameter w changes significantly with the exponent α but not much with η, except in the range $|\eta - \eta_1| < 2.5/\beta$. That is, the scalar field comes to dominate the universe, presumably for the second time, when $0 < \eta - \eta_{\rm late} \lesssim 2.5/\beta$, and it gradually becomes a constant value, $w \simeq$ const, when $\eta > \eta_{\rm late} + 2.5/\beta$. At late times, one finds $-1 \lesssim w \le -1/3$, depending upon the value of α, see Fig. 3. Whether the scalar field σ dominates the future evolution of the universe depends on energy density of matter system.

In a known form for the GB coupling, derived from the heterotic string theory, $\xi(\sigma) \sim \ln 2 - \frac{2\pi}{3}\cosh\sigma$ [15, 23]. It follows that $f(\sigma) \sim c_0 + c_1\,(\mathrm{e}^{\sigma} + \mathrm{e}^{-\sigma})$. For $\sigma > 0$, only the first two terms are relevant. To this end, we slightly modify the ansatz for $u(\sigma)$, namely $u(\sigma) \equiv u_0(1 + u_1\,\mathrm{e}^{\alpha\eta})$ with $\alpha\eta < 0$, which may be appropriate at some intermediate epoch. Note that $R/2\kappa^2 = 3(2H^2 + \dot{H})/\kappa^2$ and $f(\sigma)\mathcal{G} = 3u(H^2 + \dot{H})/\kappa^2$. We demand $0 < |u| \ll 2$, so that the GB contribution to the action is only sub-dominant. The solution for h is modified as

$$h = -(\hat{\beta} + \beta)\left[1 + \left(\frac{4+\sqrt{13}}{\hat{\beta}+\beta} - 1\right)\sqrt{1 + u_1\,\mathrm{e}^{\alpha\eta}}\,K(\eta)\right] \tag{21}$$

where $K(\eta) \equiv (L_{P_+} + cL_{Q_+})/(L_{P_-} + cL_{Q_-})$ with $L_{P_\pm}$ ($L_{Q_\pm}$) being associated Legendre functions of first (second) kind, $L_{P(Q)}\left(\frac{2\beta}{\alpha} \pm \frac{1}{2}, \frac{2\sqrt{13}}{\alpha}, \sqrt{1 + u_1\mathrm{e}^{\alpha\eta}}\right)$, and c an integration constant. For $|\eta - \eta_1| \gg 0$ and $\alpha < 0$, so $u(\sigma) \to u_0$, we find $h \simeq -4 + \sqrt{13}\tanh\sqrt{13}(\eta - \eta_1) \simeq -0.3944$ and hence $w_{DE} \simeq -0.7370$. This value of w is realized only at the asymptotic future, where $u(\sigma)$ attains a small (constant) value, while its present value, w_{DE}^0, may be less than w_{DE}. In the presence of matter coupling, $\Omega_m \simeq 0.3$, one defines an effective EOS parameter, w_{eff}, whose value can be greater than w_{DE}^0.

It is generally valid that the transformation from the string frame to the Einstein frame introduces couplings between standard model fields and massless degrees of freedom, such as, dilaton ϕ (a spin-0 partner of spin-2 graviton) [24]. Simple arguments such as the absence of ghost, thereby guaranteeing the stability of the field theory, would suffice to rule out a wide class of scalar-tensor models. In our model, we have assumed, rather implicitly, that dilaton is either constant or it is extremized, $\frac{\mathrm{d}\lambda(\phi)}{\mathrm{d}\phi} \simeq 0$, so the effects coming from terms like $\lambda(\phi)\mathcal{G}$ are negligible. Only the dynamical field in the model is the modulus σ, which is expected to have a non-zero mass, i.e., a non-zero vacuum expectation value. In string theory context [15], there is no coupling between the field σ and the Einstein-Hilbert term. And there is no direct coupling of the field σ to matter, in view of precise veri-

fications of the weak equivalence principle. Nevertheless, it is useful to assume that the running of the field σ is negligibly small at late times, namely [25, 26]

$$H^2 \dot{\sigma} \frac{df(\sigma)}{d\sigma} < 10^{-3}, \tag{22}$$

so as to avoid a possible conflict with the time-variation of Newton's constant under Newtonian approximation.

In conclusion, we find that the dark energy hypothesis fits into a low energy gravitational action where a scalar field is coupled to the curvature squared terms in the Gauss-Bonnet combination. It is established that a GB-scalar coupling can play an important and interesting role in explaining both the early and late-time acceleration of the universe, with singularity-free solutions. An important feature of our solutions is also that the effective potential (or dark energy) can naturally relax to a sufficiently low value, $\Lambda \sim 10^{-120} M_P^4$, after > 125 e-folds of expansion.

It is shown that one may cross the cosmological constant/phantom barrier, $w = -1$, only by allowing the parameter α, coming from the GB coupling, such that the Hubble rate grows with proper time, or by allowing a non-canonical scalar, $\gamma < 0$ (phantom field).

A number of follow-up studies may be devised including: a fit to determine how well the potential may be tuned to the present value of the cosmological constant, and how well slow roll parameters fit the spectral index in an imprint on the cosmic microwave background anisotropy and mass power spectrum. Some of the results appear elsewhere [27]. It is possible to generalize the model studied here to a system of two scalar fields, by introducing a dilaton field ϕ [28].

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[1] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1.

[2] P.J.E. Peebles and B. Ratra, Rev. Mod. Phys. **75** (2003) 559, astro-ph/0207347;
T. Padmanabhan, Phys. Rept. **380** (2003) 235;
E.J. Copeland, M. Sami and S. Tsujikawa, hep-th/0603057.

[3] P.J.E. Peebles and B. Ratra, Astrophys. J. **325** (1988) L17; C. Wetterich, Nucl. Phys. B **302** (1988) 668.

[4] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, Phys. Rept. **215** (1992) 203.

[5] A.D. Linde, Phys. Lett. B **108** (1982) 389.

[6] I. Antoniadis, E. Gava and K.S. Narain, Phys. Lett. B **283** (1992) 209;
I. Antoniadis, E. Gava and K.S. Narain, Nucl. Phys. B **383** (1992) 93.

[7] C.L. Bennett *et al.*, Astrophys. J. Suppl. **148** (2003) 1;
D.N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148** (2003) 175;
A.G. Riess *et al.* [Supernova Search Team Collaboration], Astrophys. J. **607** (2004) 665.

[8] R.R. Caldwell, Phys. Lett. B **545** (2002) 23.

[9] S. Capozziello, Int. J. Mod. Phys. D **11** (2002) 483;
S.M. Carroll, V. Duvvuri, M. Trodden and M.S. Turner, Phys. Rev. D **70** (2004) 043528.

[10] S. Capozziello, V.F. Cardone, S. Carloni and A. Troisi, Int. J. Mod. Phys. D **12** (2003) 1969; S. Nojiri and S.D. Odintsov, Phys. Rev. D **68** (2003) 123512;
G. Allemandi, A. Borowiec and M. Francaviglia, Phys. Rev. D **70** (2004) 103503.

[11] G.R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485** (2000) 208;
C. Deffayet, G.R. Dvali and G. Gabadadze, Phys. Rev. D **65** (2002) 044023.

[12] P.K. Townsend and M.N.R. Wohlfarth, Phys. Rev. Lett. **91** (2003) 061302, hep-th/0303097;
N. Ohta, Phys. Rev. Lett. **91** (2003) 061303;
S. Roy, Phys. Lett. B **567** (2003) 322;
C.M. Chen, P.M. Ho, I.P. Neupane and J.E. Wang, JHEP **0307** (2003) 017, hep-th/0304177;
C.M. Chen, P.M. Ho, I.P. Neupane, N. Ohta and J.E. Wang, JHEP **0310** (2003) 058, hep-th/0306291;
I.P. Neupane, Class. Quant. Grav. **21** (2004) 4383, hep-th/0311071.

[13] I.P. Neupane and D.L. Wiltshire, Phys. Lett. B **619** (2005) 201, hep-th/0502003;
I. P. Neupane and D. L. Wiltshire, Phys. Rev. D **72** (2005) 083509, hep-th/0504135.

[14] C.G. Callan, D. Friedan, E.J. Martinec and M.J. Perry, Nucl. Phys. B262 (1985) 593; E.S. Fradkin and A.A. Tseytlin, Phys. Lett. B158 (1985) 316; D.J. Gross and J.H. Sloan, Nucl. Phys. **B291** (1987) 41.

[15] I. Antoniadis, J. Rizos and K. Tamvakis, Nucl. Phys. B **415** (1994) 497.

[16] T. Damour and A.M. Polyakov, Gen. Rel. Grav. **26** (1994) 1171, gr-qc/9411069.

[17] J.C. Fabris, A.M. Pelinson and I.L. Shapiro, Nucl. Phys. B **597** (2001) 539 [Erratum-ibid. B **602** (2001) 644].

[18] S. Nojiri and S.D. Odintsov, Phys. Lett. B **631** (2005) 1.

[19] M. Sami, A. Toporensky, P.V. Tretjakov and S. Tsujikawa, Phys. Lett. B **619** (2005) 193;
G. Calcagni, S. Tsujikawa and M. Sami, Class. Quant. Grav. **22** (2005) 3977, hep-th/0505193.

[20] S. Nojiri, S.D. Odintsov and M. Sasaki, Phys. Rev. D **71** (2005) 123509, hep-th/0504052.

[21] A.A. Starobinsky, Phys. Lett. B **91** (1980) 99.

[22] R.R. Caldwell, M. Kamionkowski and N.N. Weinberg, Phys. Rev. Lett. **91** (2003) 071301.

[23] R. Easther and K.i. Maeda, Phys. Rev. D **54** (1996) 7252.

[24] T. Damour, F. Piazza and G. Veneziano, Phys. Rev. Lett. **89** (2002) 081601, gr-qc/0204094.

[25] G. Esposito-Farese, AIP Conf. Proc. **736** (2004) 35, gr-qc/0409081.

[26] L. Amendola, C. Charmousis and S.C. Davis, hep-th/0506137.

[27] I.P. Neupane and B.M.N. Carter, hep-th/0512262, JCAP, in press.

[28] I.P. Neupane, hep-th/0602097.